Open problems on promotion, cyclic sieving, and trees

Bruce Sagan
Michigan State University
www.math.msu.edu/~sagan

September 11, 2025

Let

$$[n] = \{1, 2, \ldots, n\}.$$

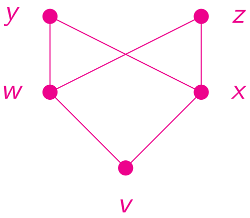
Let $(P, \leq)$ be a finite poset (partially ordered set). If $\#P = n$ then a *natural labeling* of P is a bijection $L : P \to [n]$ such that

$$x \leq y \text{ in } P \implies L(x) \leq L(y).$$

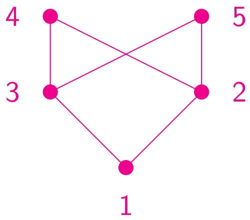
Let

$$\mathcal{L}(P) = \{L \mid L \text{ is a natural labeling of } P\}.$$

Ex. $P =$



$L =$



$\in \mathcal{L}(P)$

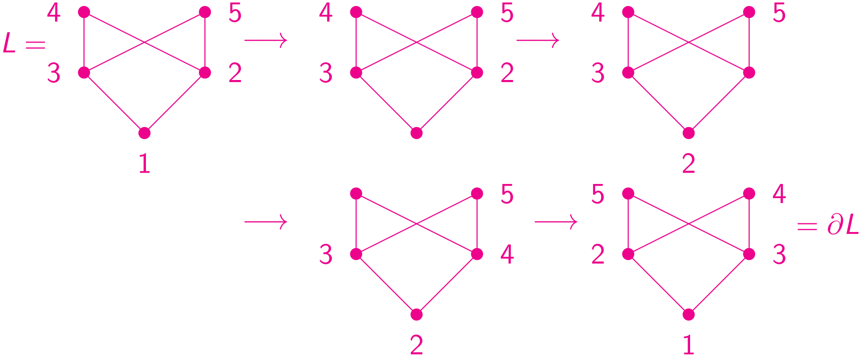
Promotion is the bijection $\partial : \mathcal{L}(P) \to \mathcal{L}(P)$ defined as follows.

(1) Given $L \in \mathcal{L}(P)$, remove the label 1 from its element x.

(2) Let y have minimum label $L(y) = \ell$ among all covers of x and move ℓ from y to x.

(3) Iterate (2) until some maximal element z becomes unlabled. Subtract 1 from all labels and let $L(z) = \#P$ to form ∂L.

Ex.



Theorem (Haiman, 1992)

Promotion acting on $\mathcal{L}([m] \times [n])$ has order $o(\partial) = mn$.

Let $G = \langle g \rangle$ be a finite group with generator g acting in a finite set S. Element $h \in G$ has *fixed point set*

$$S^h = \{s \in S \mid hs = s\}.$$

Let $f(q)$ be a polynomial in q. The triple $(S, G, f(q))$ exhibits the *cyclic sieving phenomenon, (CSP),* if for every $h \in G$ we have

$$\#S^h = f(\omega)$$

where ω is a root of unity such that $o(h) = o(\omega)$.

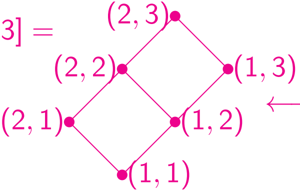
Poset $[m] \times [n]$ corresponds to a Young diagram of rectangular shape (n^m). So $L \in \mathcal{L}([m] \times [n])$ becomes a standard Young tableau T of this shape.

Theorem (Rhoades, 2010)

The triple $(\mathcal{L}([m] \times [n]), \langle \partial \rangle, f_{m,n}(q))$ exhibits the CSP where $f_{m,n}(q)$ is a q-analogue of the hook formula for the shape (n^m).

Ex.

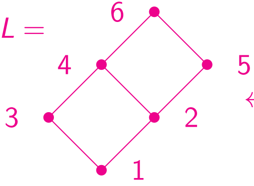
$[2] \times [3] =$



$\longleftrightarrow (3^2) =$

(1, 1)	(1, 2)	(1, 3)
(2, 1)	(2, 2)	(2, 3)

$L =$



$\longleftrightarrow \quad T =$

1	2	5
3	4	6

A *tree* is a poset T such that the Hasse diagram of T is a graph-theoretic tree. The *comb, C_n,* is obtained by adding a maximal element to each element of $[n]$.

Theorem (Kimble, S, St. Dizier, 2025)

If $\mathcal{O}$ is an orbit of ∂ acting on $\mathcal{L}(C_n)$ then

$$\#\mathcal{O} = (2n-1)(2n-3)(2n-5)\cdots(2n-2\lceil n/2\rceil+2).$$

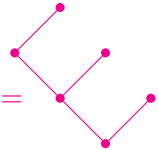
Define the *hooklength* of $v \in T$ as: $\qquad h_v = \#\{w \in T \mid w \geq v\}.$

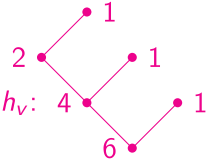
$$\therefore \#\mathcal{L}(T) = \frac{n!}{\prod_{v\in T} h_v} \text{ where } n = \#T. \qquad (1)$$

Problem

Find a natural $g_n(q)$ so that $(\mathcal{L}(C_n), \langle\partial\rangle, g_n(q))$ exhibits the CSP and similar polynomials work with trees having different orbit sizes. The usual q-analogue of (1) does not work!

Ex.

$C_3 =$ 



References

1. Mark D. Haiman. Dual equivalence with applications, including a conjecture of Proctor. *Discrete Math.*, 99(1–3): 79–113, 1992.

2. J. Kimble, B. Sagan, and A. St. Dizier. K-promotion on m-packed labelings of posets. Preprint arXiv:2508.09305, 1–30, 2025.

3. V. Reiner, D. Stanton, and D. White. The cyclic sieving phenomenon. *J. Combin. Theory Ser. A*, 108(1):17–50, 2004.

4. B. Rhoades. Cyclic sieving, promotion, and representation theory. J. Combin. Theory Ser. A 117 (2010), no. 1, 38–76.

5. M. P. Schützenberger. Promotion des morphismes d'ensembles ordonnés. *Discrete Math.*, 2:73–94, 1972.

THANKS FOR LISTENING!